

# Quantizing the Line Element Field.

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## Abstract

A metric with signature  $(-+++)$  can be constructed from a metric with signature  $(++++)$  and a double-sided vector field called the line element field. Some of the classical and quantum properties of this vector field are studied.

## Contents

1	Introduction.	1
2	Curvature.	2
3	Quantization.	5
4	Conclusion.	7

## 1 Introduction.

The difference between a Lorentzian and a positive definite metric can be expressed as a double-sided vector field  $U$  called the line element field. This is done in [4]p.38, but there only the ratio of vectors in the two spaces is considered; this appears to be the only reference on the line element field and it is from it that the nomenclature is taken. Thus the study of fields or extended objects in Lorentzian spacetime is reduced to the study of the same object in a positive definite space and the study of the corresponding line element field. In particular this can be done for gravity, where the positive definite action is sometimes called the Euclidean action [3]. Things not looked at here include: *firstly* any relationship to analytic continuation, whether for quantum field theory on curved spacetime or for the energy condition [4]p.89, *secondly* any classical or quantum detailed mechanism or perturbation whereby a positive definite space could change to a Lorentzian spacetime, for example in the early universe, *thirdly* the connection with the Kubo-Martin-Schwinger [2] condition where the transformation  $\tau \rightarrow it$  has thermal properties, *fourthly* a quantized line element field might fluctuate, this fluctuation could be thought of in terms of the tetrad rather than the metric, leading to fluctuating null cones, compare Penrose [6], *fifthly* any comparison with the Toll [12] - Scharnhorst [10] effect, where fluctuations in the quantum electrodynamical vacuum cause fluctuations in the speed of light, *sixthly* any comparison with the average size of these fluctuations, compare Ellis *et al* [1] and Yu and Ford [13], *seventhly* not only can the difference between the two signatures be thought of as a vector field, also the difference between tensors constructed from the resulting metrics is tensorial, the Bianchi identities will also differ by a tensorial object constructed from the line element field and this gives another way of investigating conservation laws for the two signatures, compare [5].

In §2 some examples of positive definite metrics are presented and how to change their signature via a vector field is shown; this is successively generalized to vanishing shift metrics and then the general theory, next the first derivatives are studied, and expressed in terms of a contorsion tensor. The second derivatives of the metric are governed by the Riemann tensor which can be expressed by independent terms in the contorsion and Christoffel connection. In §3 the Einstein-Hilbert action is decomposed into a positive definite part and a line element field part, the line element field part is varied with respect to both  $U$  and  $\dot{U}$ . The variation with respect to  $\dot{U}$  gives the momentum. Quantization is implemented by replacing this momentum by a differential operator to give a modified Klein-Gordon equation. Then the lowest order approximation to the modified Klein-Gordon equation is calculated, and the wavefunction is calculated for some specific spacetimes.

Notation used includes the bracket notation of [4]p.20

$$2V_{(a,c)} = V_{a,c} + V_{c,a} = 2V_{(a;c)} + 2\{^e_{ca}\}V_e, \quad 2V_{[a,c]} = 2V_{[a;c]} = V_{a,c} - V_{c,a}, \quad (1)$$

the scalars constructed from the expansion and vorticity

$$\theta \equiv \theta^a_{;a}, \quad \omega^2 \equiv \omega^{ab}\omega_{ab}, \quad \sigma^2 \equiv \sigma^{ab}\sigma_{ab}, \quad (2)$$

and vector fields

$U_a$  for a general vector,

$V_a$  for a normalization of this to  $\pm 1$ ,

$W_a$  for a specific vector. (3)

## 2 Curvature.

For a given positive definite metric  ${}^{-+}p_{ab}$  and vector field  ${}^{++}U_a$ , one can construct a Lorentzian spacetime with covariant metric

$${}^{-+}g_{ab} = -2 \frac{{}^{++}U_a {}^{++}U_b}{{}^{++}U^2} + {}^{++}p_{ab}, \quad (4)$$

This can be illustrated using the positive definite Schwarzschild metric

$${}^{++2}ds^2 = + \left(1 - \frac{2m}{r}\right) dx_0^2 + \left(1 - \frac{2m}{r}\right)^{-1} + r^2 d\Sigma_2^2, \quad {}^{++}W_a = \sqrt{1 - \frac{2m}{r}}, \quad {}^{++a}W_a = +1, \quad (5)$$

or the positive definite Robertson-Walker metric

$${}^{++2}ds^2 = + dx_0^2 + R^2 d\Sigma_3^2, \quad {}^{++}W_a = (1, 0), \quad (6)$$

then using 4 the spacetime metric is recovered. Instead of  $U$  it is often convenient to work with the unit vector

$$V_a = \frac{U_a}{\sqrt{\pm U^2}}, \quad U^2 = U^a U_a, \quad (7)$$

There is a problem of what the contravariant form of  ${}^{-+}g$ ,  $p$  &  $U$  should be. Say we are given a positive definite space with shift-free metric and vector field

$$\begin{aligned} {}^{++}p_{ab} &= (p^2, p_{ij}), & {}^{++}p^{ab} &= \left(\frac{1}{p^2}, p^{ij}\right), & \det({}^{++}p_{ab}) &= p \det(p_{ij}), \\ {}^{++}W_a &= (p, 0), & {}^{++a}W_a &= \left(\frac{1}{p}, 0\right), \end{aligned} \quad (8)$$

where for simplicity there are no cross terms  $(\tau, i)$  terms, as that would require either  $W_a$  or  $W^a$  to be no longer one component. Now one can construct a Lorentzian spacetime with covariant metric

$$\bar{g}^+_{ab} = (-p^2, p_{ij}), \quad \det(\bar{g}^+_{ab}) = -\det(\bar{p}^{++}_{ab}). \quad (9)$$

Consistency seems to require

$$\bar{g}^{+ab} = -2 \bar{V}^{+a} \bar{V}^{+b} + \bar{p}^{+ab} = \left( -\frac{1}{p^2}, p^{ij} \right), \quad (10)$$

note that only cross terms in  $V$  occur so perhaps  $\bar{V}^+$  could have been used, 17 shows that this is not the case. Taking  $\bar{V}^+_a = \bar{V}^{++}_a$  and raising using this metric

$$\bar{W}^{+a} = \left( -\frac{1}{p}, 0 \right), \quad \bar{W}^{+2} = -1, \quad (11)$$

so that  $\bar{V}^+$  is a timelike vector. Similiarly taking  $\bar{p}^{+ab} = \bar{p}^{++ab}$  and raising indices using 10 gives

$$\bar{p}^{+ab} = \bar{p}^{++ab}. \quad (12)$$

Some products using the above tensors are

$$\begin{aligned} \bar{g}^{+c}_c &\equiv \bar{g}^{+ab} \bar{g}^{+}_{bc} = \bar{p}^{+a}_c - 2 \bar{V}^{+a} \bar{V}^{+}_c, & \bar{p}^{+ab} \bar{g}^{+}_{bc} &= 2 \bar{V}^{+a} \bar{V}^{+}_c + \bar{g}^{+a}_c, \\ -\bar{V}^{+}_a &= \bar{V}^{+b} \bar{g}^{+}_{bc}, & \bar{V}^{+a} &= \bar{V}^{+c} \bar{g}^{+a}_c = -\bar{V}^{+}_c \bar{p}^{+ab}, & \bar{p}^{+ac} \bar{V}^{+}_c &= -\bar{V}^{+a}. \end{aligned} \quad (13)$$

Collecting this together consistency requires

$$\bar{g}^{+ab} = -2 \frac{\bar{U}^{+a} \bar{U}^{+b}}{\bar{U}^{+2}} + \bar{p}^{+ab}, \quad (14)$$

and

$$\bar{V}^{+}_a = \bar{V}^{++}_a, \quad \bar{V}^{+a} = -\bar{V}^{+a}, \quad \bar{p}^{+ab} = \bar{p}^{++ab}, \quad \bar{p}^{+ab} = +\bar{p}^{++ab}. \quad (15)$$

The above system is new and slips can be made by relying on ones intuition from studying spacetimes using the projection tensor, see 18 below, or confusing the  $(-+)$  and  $(++)$  spaces. The most common of these is

$$4 = g^a_a \stackrel{?}{=} -2V_a V^a + p^a_a = -2V_a V^a + 4, \quad (16)$$

suggesting that  $V$  is null, contrary to assumption. The correct calculation is

$$4 = \bar{g}^{+ab} \bar{g}^{+}_{ab} = \left( -2 \frac{\bar{U}^{+a} \bar{U}^{+b}}{\bar{U}^{+2}} + \bar{p}^{+ab} \right) \left( -2 \frac{\bar{U}^{+a} \bar{U}^{+b}}{\bar{U}^{+2}} + \bar{p}^{+ab} \right) = \left( 4 \frac{\bar{U}^{+2}}{\bar{U}^{+2}} - 2 - 2 \right) \frac{\bar{U}^{+2}}{\bar{U}^{+2}} + 4, \quad (17)$$

which also serves as showing that  $\bar{U}^{++}$  rather than  $\bar{U}^+$  should be used for  $\bar{g}^+$  in 14. The projection tensor is defined as

$$\bar{h}^{+}_{ab} \equiv \bar{g}^{+}_{ab} + \frac{\bar{U}^{+}_a \bar{U}^{+}_b}{-\bar{U}^{+2}} = \bar{p}^{+ab} - 2 \frac{\bar{U}^{+a} \bar{U}^{+b}}{\bar{U}^{+2}} - \frac{\bar{U}^{+}_a \bar{U}^{+}_b}{\bar{U}^{+2}} = \bar{p}^{+ab} + \frac{\bar{U}^{+}_a \bar{U}^{+}_b}{\bar{U}^{+2}} = \bar{p}^{+ab} - \frac{\bar{U}^{+a} \bar{U}^{+b}}{\bar{U}^{+2}}, \quad (18)$$

where in this case the indices can be raised and lowered without change of form.

Having formed the metric the next problem is the properties of its first derivatives. To form the connection  $\Gamma$  with  $g$  in terms of the connection  $\{\}$  with  $p$  one has

$$\begin{aligned} 2 \overset{g}{\Gamma}_{abc} &\equiv g_{ba,c} + g_{ca,b} - g_{bc,a} \\ &= 2\{_{abc}\} + 4U^{-2} \left( -U_a \{_{bc}^e\} U_e - U_a U_{(b;c)} + U_b U_{[c;a]} + U_c U_{[b;a]} \right) \\ &\quad + 4U^{-4} U_e \left( 2U_a U_{(b} U_{.c)}^e - U_b U_c U_{.a}^e \right), \end{aligned} \quad (19)$$

where the bracket notation 1 is used and the covariant derivatives on the rhs of 20 are formed with  $p$ . Raising with the metric 10

$$\Gamma_{bc}^a \equiv \overset{-+}{g}^{ad} \Gamma_{dbc}, \quad (20)$$

so that

$$\Gamma_{bc}^a = \{_{bc}^a\} + L^a_e \{_{bc}^e\} + K_{bc}^a, \quad (21)$$

it is found that  $L = 0$  implying that the system is covariant. From 21 there is the relation between the covariant derivatives.

$$\overset{++}{V}_a \overset{++}{;}^+ b = \overset{++}{V}_a \overset{++}{;}^{++} b - K_{ab}^c \overset{++}{V}_c, \quad (22)$$

The contorsion tensor  $K$  is

$$\begin{aligned} 2K_{bc}^a &\equiv +8U^{-2} U_{(b} J_{c)}^a + 4U^{-2} U^a U_{(bc)} - 2U^{-4} M_{bc}^a, \\ J_{ab} &\equiv U_{[a;b]} - 2U^{-2} U_b U^c U_{[a;c]}, \\ M_{bc}^a &\equiv (U^2)^a U_b U_c + (U^2)_b U^a U_c + (U^2)_c U^a U_b - 2U^{-2} U^a U_b U_c U^e (U^2)_e, \end{aligned} \quad (23)$$

there is no term in the accelerations  $\dot{U}^a$ ,  $\dot{U}^b$  or  $\dot{U}^c$ ,  $M$  vanishes if  $U$  is a constant vector. Some properties of the contorsion  $K$  are

$$\begin{aligned} K_{ac}^a &= 0, \quad K_{ba}^a = 0, \quad K_{bc}^a = K_{cb}^a, \quad U^b U^c K_{bc}^a = 0, \\ U_a K_{bc}^a &= 2U_{bc} - 2U^{-2} (U_{(b} \dot{U}_{c)} + U_e U_{(b} U_{.c)}^e) + 2U^{-4} U_b U_c U_e \dot{U}^e, \end{aligned} \quad (24)$$

the dot being formed with the  $p$  covariant derivatives; such systems involving a connection and a contorsion occur repeatedly in the study of curvature; for instance in geometries involving torsion and/or metricity such as the geometries of Weyl and Schouten [11], the study of a conformal factor, and the study of weak metric perturbations [9]. Alternatively the contorsion can be expressed in terms of the decomposed vector field for a  $(-+)$  Lorentzian spacetime define  $\theta$  as in 2 and

$$\omega_{ab} \equiv h_a^c h_b^d V_{[c;d]}, \quad \theta_{ab} \equiv h_a^c h_b^d V_{(c;d)}, \quad \sigma_{ab} \equiv \theta_{ab} - \frac{1}{3} \theta h_{ab}, \quad \dot{X}_{abc\dots} \equiv V^e X_{abc\dots e}, \quad (25)$$

which allow the covariant derivative of a  $(-+)$  spacetime to be decomposed

$$U_{a;b} = \theta_{ab} + \omega_{ab} + U^{-2} U_b \dot{U}_a + U^{-2} U_a U^e U_{eb} - U^{-4} U_a U_b U^e \dot{U}_e, \quad (26)$$

choosing a constant vector field this reduces to [4]eq.4.17. Now the equations 24 are in the  $(++)$  space and 25 are in a  $(-+)$  spacetime; they can be related using the projection tensor 18. The projections of the covariant derivative are

$$\begin{aligned} h_a^c h_b^d U_{(c;d)} &= U_{(a;b)} - U^{-2} U_{(a} \dot{U}_{b)} - \frac{1}{2} U^{-2} U_{(a} (U^2)_{b)} + \frac{1}{2} U^{-4} U_a U_b (\dot{U}^2) \\ h_a^c h_b^d U_{[c;d]} &= U_{[a;b]} - U^{-2} \dot{U}_{[a} U_{b]} - \frac{1}{2} U^{-2} U_{[a} (U^2)_{b]} \end{aligned} \quad (27)$$

To transfer these quantities to the  $(++)$  space, the projection tensor 18 shows that it is only necessary to note that the negative quantity  $U^2 = \overset{-+}{U}_a \overset{-+}{U}^a < 0$  is changed to the positive quantity

$U^2 = U_a^{++} U^{++a} > 0$ , and also that the covariant derivative in the expansion is changed using 22.  $U_e K_{ab}^e$  is  $-2$  times  $\theta_{ab}$  so that the sign of  $\theta_{ab}$  in the  $(++)$  space is the negative of the form in 27 and from 25. In particular

$$-\theta = U_{.;a}^a - \frac{(\dot{U}^2)}{U^2}. \quad (28)$$

Using 24 and 27 the contorsion tensor is found to be

$$J_{ab} = \omega_{ab} - \frac{\dot{U}_{(a} U_{b)}}{U^2} + \frac{U_{(a} (U^2)_{b)}}{2U^2}, \quad K_{bc}^a = \frac{4}{U^2} U_{(b} \omega_{c)}^a - \frac{2}{U^2} U^a \theta_{bc} - 2\sqrt{U^2} \left( \frac{\dot{U}^a}{\sqrt{U^2}} \right) \frac{U_b U_c}{U^4}. \quad (29)$$

The second derivatives are governed by the Riemann tensor which is, c.f.eq.111[9]

$$\overset{g}{R}{}^a{}_{bcd} = \overset{p}{R}{}^a{}_{bcd} + 2K_{.[d|b|;c]}^a + 2K_{.[c|e|}^a K_{.d]b}^e. \quad (30)$$

### 3 Quantization.

Contracting the expression for the Reimann tensor 30 and using 9 the Einstein-Hilbert Lagrangian is

$$\mathcal{L}_H = \sqrt{-g} \overset{g}{R} = \sqrt{\det(p_{ab})} \left[ \overset{p}{R} + 2K_{.[b|.;a]}^a + 2K_{.[a|e|}^a K_{.b]}^{eb} \right] = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3. \quad (31)$$

$\mathcal{L}_1$  is the positive definite action, sometimes called the Euclidean action which has previously been studied,  $\mathcal{L}_2$  &  $\mathcal{L}_3$  are new and easiest to describe in terms of the decomposed vector quantities 25

$$\begin{aligned} \mathcal{L}_2 &= -\frac{2}{U^2} (\dot{\theta} - \theta^2) - 2 \left( \frac{1}{\sqrt{U^2}} \left( \frac{U^a}{\sqrt{U^2}} \right)^\circ \right)_a \equiv l_1 + l_2 + l_3, \\ \mathcal{L}_3 &= \frac{4\omega^2}{U^2} - \frac{4}{U^4} \left[ \left( \frac{U^a}{\sqrt{U^2}} \right)^\circ U_a \right]^2 \equiv l_4 + l_5, \quad \omega^2 \equiv \omega^{ab} \omega_{ab}, \end{aligned} \quad (32)$$

expanding  $l_5 = 0$ . Varing with respect to  $U$ ,

$$\begin{aligned} \frac{\delta l_1}{\delta U^c} &= -2U^{-2} \theta_c + 2 \left( (U^{-2} U^e U^2)_e \right)_c - 2U^{-2} (U^{-2} U^e)_e (U^2)_c \\ &\quad - 4U_c \left( U^{-2} (-^2 U^f)_f U^e \right)_e - 4U^{-4} (U^{-2} U^e)_e (U^2)^\circ U_c, \\ \frac{\delta l_2}{\delta U^c} &= 4 (U^{-2} \theta)_c + 4U^{-4} \theta (U^2)_c - 8U_c (U^{-4} \theta U^e)_e - 8U^{-6} (U^2)^\circ U_c, \\ \frac{\delta l_4}{\delta U^c} &= -8 (U^{-2} \omega^{ce})_e + 4U^{-2} \omega^{ce} \left( 2\dot{U}_e - U^2 (U^2)_e \right), \\ \frac{\delta l_3}{\delta U^c} &= \frac{\delta l_5}{\delta U^c} = 0. \end{aligned} \quad (33)$$

Varing with respect to  $\dot{U}^c$

$$\frac{\delta l_1}{\delta \dot{U}^c} = -\frac{4\theta U_c}{U^4}, \quad \frac{\delta l_2}{\delta \dot{U}^c} = +8 \frac{\theta U_c}{U^4}, \quad \frac{\delta l_3}{\delta \dot{U}^c} = \frac{\delta l_4}{\delta \dot{U}^c} = \frac{\delta l_5}{\delta \dot{U}^c} = 0. \quad (34)$$

Therefore

$$\Pi_a = \frac{\delta}{\delta \dot{U}^a} (l_1 + l_2) = \frac{4\theta U_a}{U^4}. \quad (35)$$

This equation is not fully invertable  $U_a = f(\Pi) \Pi_a$ , but is partially invertible  $U_a = f(\Pi, U) \Pi_a$ ,

$$\frac{U_a}{\sqrt{U^2}} = \frac{\Pi_a}{\sqrt{\Pi^2}}, \quad (36)$$

and partially invertible  $U_a = f(\Pi, \theta)\Pi_a$ ,

$$U_a = (4\theta)^{\frac{1}{3}}\Pi^{-\frac{4}{3}}\Pi_a. \quad (37)$$

35 gives the constraint

$$\lambda = \Pi_c \Pi^c - \frac{16\theta^2}{U^6}. \quad (38)$$

This is the only constraint so that quantization can be achieved via

$$\Pi_a \rightarrow -i\hbar \nabla_a \quad (39)$$

with  $U$  and hence  $\theta$  remaining unchanged. Plancks constant  $\hbar$  is of the same dimensions as action, explicitly  $Mass \times Lenght^2 \times Time^{-1}$ , so that 39 has introduced a "mass" into the system. Applying 39 to the constraint gives a modified Klein-Gordon equation

$$\lambda\psi = -\hbar^2 \left( \square + \frac{16\theta^2}{\hbar^2 U^6} \right) \psi = 0. \quad (40)$$

Defining

$$S \equiv -i\hbar \ln \psi \quad (41)$$

the modified Klein-Gordon equation 40 becomes

$$-i\hbar S_{;a}^s + S^a S_a - \frac{16\theta^2}{U^6} = 0, \quad (42)$$

expanding in terms of  $\hbar$  using

$$S_a = \Pi_a + \hbar\epsilon_a + \mathcal{O}(\hbar^2), \quad (43)$$

the  $\hbar^0$  term is just the constraint 38, the  $\hbar^1$  term is

$$-i\Pi_{;a}^a + 2\epsilon_a \Pi^a = 0, \quad (44)$$

For  $\theta = 0$ , the lagrangians  $l_1$  and  $l_2$  vanish as does  $\Pi$ , so that to lowest order  $\hbar^0$ ,  $S_a = 0$ , implying that the wavefunction  $\psi$  is a constant to lowest order, thus for  $\theta = 0$  the wavefunction has no dynamical information corresponding to the classical theory.

$U$  remains unchanged during quantization, but once a solution  $\epsilon$  to 43 is known, one would hope to be able to calculates the  $\hbar^1$  order correction to  $U$  and hence  $g$ . There is a problem with trying this, as  $\Pi$  is only partially invertible 36 & 37 this cannot be done without an additional assumption. Here this assumption is that  $\theta$  remains negligible to order  $\hbar^1$  in the quantum theory, then it is possible to find the correction to  $U$  from 37, denoting the quantum quantities with a "\*"  $U$  becomes

$$U_a^* = (4\theta)^{\frac{1}{3}} S^{\frac{4}{3}} S_a. \quad (45)$$

Substituting for  $S$  using 43 and expanding

$$U_a^* = U_a + \frac{\hbar}{4} \frac{U^4}{\theta} \left( \epsilon_a - \frac{8}{3} \frac{U_c \epsilon^c}{U^2} U_a \right) + \mathcal{O}(\hbar^2). \quad (46)$$

It is now possible to investigate whether the assumption that  $\theta$  is negligible by noting

$$\begin{aligned} -\theta^* &\equiv U_{;a}^{*a} - U^{*-2} (U^{*2})^\circ = -\theta \\ &+ \frac{\hbar}{4} \left[ \frac{U^4}{\theta} (\epsilon^a - \frac{8}{3} U^{-2} U^c \epsilon_c U^a) \right]_a + \frac{5\hbar}{6} \left[ U^{-2} \left( \frac{U^4 \epsilon_a U^a}{\theta} \right)^\circ - \frac{(U^2)^\circ \epsilon_a U^a}{\theta} \right] + \mathcal{O}(\hbar^2). \end{aligned} \quad (47)$$

Substituting for  $U^*$  the change in the metric is

$$g_{ab}^* = g_{ab} + \frac{\hbar}{\theta} (U_c \epsilon^c U_a U_b - U^2 U_{(a} \epsilon_{b)}) + \mathcal{O}(\hbar^2). \quad (48)$$

The change in the metric can also be directly calculated from the wavefunction

$$g_{ab}^* - g_{ab} - 2U^{-2}U_a U_b = -2U^{-2}U_a^* U_b^* = -2S^{-\frac{4}{3}}S_a S_b = 2\hbar^2(-i\hbar \ln \psi)^{-\frac{4}{3}}\psi^{-2}\psi_a \psi_b. \quad (49)$$

The modified Klein-Gordon equation 40 can be studied for particular examples, for example in Robertson-Walker spacetime 6 it is

$$\psi_{00} + 3\frac{R_0}{R}\psi_0 + \frac{144}{\hbar^2}\frac{R_0^2}{R^2}\psi - \frac{l(l+2)}{R^2}\psi = 0, \quad (50)$$

where the last term comes from decomposing the "spatial" part into spherical harmonics [9]§4.1. For the Milne universe, which is flat when  $k = -1$  [8],  $R = t$  and 50 has solution

$$\psi = At^{-1 \pm \sqrt{1-\alpha}}, \quad \alpha = \frac{144}{\hbar^2} - l(l+2), \quad (51)$$

so that  $g^*$  is of the form  $f\hbar^2/t^2$ . For deSitter space [4]p.125,  $R = \exp(\sqrt{\Lambda/3}t)$  and when  $l = 0$  50 has solution

$$\psi = A \exp\left(\frac{1}{2}\sqrt{3\Lambda}(-1 \pm \sqrt{1-16/\hbar^2})t\right), \quad (52)$$

so that  $g^*$  is of the form  $f\hbar^2$ .

## 4 Conclusion.

The transformation between some specific positive definite spaces and Lorentzian spacetime can be achieved via a line element field 5 6. This can be generalized to shift-free and then arbitrary metrics; there is a problem of what the contravariant form of the metric should be, consistency requires 14. Once the Lorentzian metric has been expressed in terms of a positive definite metric and a vector field it is possible to study first derivatives. In 21  $L = 0$  so that the Lorentzian connection splits up into the positive definite connection and a contorsion term constructed from the line element field  $U$ ; this is similar to many other systems, such as those involving Schouten [11] geometries and weak perturbations [9]; that  $L = 0$  perhaps is not surprising as the decomposition of the Lorentzian metric is covariant. The form of the contorsion tensor 24 involves a lot of terms when expressed solely in terms of  $U$ , however using rotation and shear it takes a simpler form 29. Covariant derivatives in Lorentzian spacetime and the positive definite space are equated via 22, so that the difference is expressible as  $V_c K_{ab}^c$  and this is proportional to the expansion of  $U$ , changing spaces has the effect of changing the sign of the expansion. Second derivatives of the line element field  $U$  can be calculated once the contorsion  $K$  is known via 30.

To quantize the system it is necessary to have more information, such as what the Lagrangian and momentum are. Here the vacuum-Einstein-Hilbert Lagrangian is assumed 31, and further that it can be decomposed into a positive definite part and a line element field part which have well-defined and useful variations. Variations with respect to the metric and the line element field 33 can be done, however of more use is variation with respect to  $\dot{U}$  which is taken to give a momentum 35; variations with respect to dotted quantities also occur in the quantization of perfect fluids [7]. The momentum obeys the constraint 38. The two-sided nature of  $U$ , the Lorentzian metric is invariant under  $U \rightarrow -U$ ; and the ability to use  $U$  of different sizes to construct the Lorentzian metric do not seem to lead to further constraints. Quantization can be achieved via 39. The problem with this is that it introduces a mass into the system. The classical theory is just a theory involving length and time, however Planck's constant has dimensions  $Mass \times Length^2 \times Time^{-1}$ , so that using it in quantization introduces new quantities of dimension  $Mass$ . Theories, such as the vacuum-Einstein equations, involving just length and time are usually reversible, in the sense that the sign on the time coordinate can be changed and the field equations still obeyed; however this is no longer necessarily the case once quantities of dimensions of mass have been introduced, as illustrated by the fact that things fall down not up. This is not only a problem for the theory under study here, similarly using  $\hbar$  in quantization of the vacuum-Einstein equations

will introduce a mass. The specific wavefunctions 51 and 52 illustrate the above, of the two terms in the square root one is dimensionless "1" and the other is dimensionfull and proportional to  $\hbar^{-2}$ . A way of avoiding the above is to divide  $\hbar$  by the Planck mass or perhaps an arbitrary mass so that objects of dimensions of mass no longer occur in the quantum system; also by analogy with the point particle one could perhaps pre-multiply the line element field Lagrangian by an arbitrary  $m$ , but on the analysis so far such an  $m$  does not occur naturally, perhaps it might do so in an extended theory which in some way incorporates that  $U$  is not necessarily of unit size. Any given Lorentzian metric can be constructed from many different sets of a positive definite metric and a line element field. For example flat spacetime can be expressed by the Minkowski metric and this can be constructed from a diagonal metric and unit expansion free line element field; also flat spacetime can be expressed by the Milne universe 51 for which  $U$  has expansion. In the first case there is no expansion and hence no momentum or quantum theory, in the second there is with wavefunction 51. Thus it might be that "Euclidean" quantum gravity expresses the full quantum nature of a Lorentzian spacetime if the relating line element field is expansion free; however the main application of such theories is to the early universe where expansion is the most salient feature.

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